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Generalised charged static dust in relativity

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Abstract. For a static dust distribution charged in both the electric and scalar sense we prove that the sum of squares of these two types of charge densities must be either greater than or equal to the square of the mass density, when the scalar potential is a function of the electrostatic potential. In the absence of either the electrical or scalar field the equality sign holds. In general, if there is no singularity in the matter distribution the three-space is conformally Euclidean.

We show how to generate a special class of exact solutions of empty space in the presence of electrostatic and zero mass scalar fields and some properties are discussed.

Finally we present two exact spherically symmetric solutions as examples.

1. Introduction

There are some general results in the literature for electrically charged dust in equilibrium (De and Raychaudhuri 1968, Das 1962) as well as for electrovac (Weyl 1917, Majumdar 1947, Papapetrou 1947). In the work of De and Raychaudhuri it was shown that the equality of mass and charge densities follows from the field equations provided there is no singularity in the distribution. Later the result was proved in a similar way also for static dust charged in the scalar sense (Wolk *et al* 1975). Das, however, found that for dust in equilibrium charged in either the electric or scalar sense, all the field equations reduce to a single nonlinear equation.

In the present paper we have generalised all these results for a static distribution of incoherent dust, charged both in the electric and in the scalar sense, with the assumption of a functional relation between the scalar potential and the electrostatic potential. Such a functional relation implies that the two types of charge are closely related. Our results include, in fact, generalisations of even those obtained by Teixeira *et al* (1976) in the spherically symmetric case. We have further extended our calculations in the empty space containing source-free electromagnetic as well as massless scalar fields.

In § 2 we have proved that for a static dust distribution charged both in the electrical and scalar sense the sum of the squares of these two types of charge densities must be either greater than or equal to the square of the mass density. In the absence of either the electrical or scalar field, however, only the equality sign holds. For charged matter there is a unique charge-to-mass ratio such that the electrical repulsion just balances the gravitational attraction, and the system can be static. If two types of matter are present with two types of charges, the repulsions act only between equal types, whereas gravitational attraction acts between all matter.

It is therefore not surprising that the static condition yields an inequality between the charges and mass, and this paper develops the detailed mathematics of this situation. It has been further shown that in general if there is no singularity in the matter distribution the three-space (t = constant) is conformally Euclidean and the whole set of field equations reduces to a single nonlinear differential equation.

In § 3 we have shown how to generate a special class of exact solutions for empty space in the presence of electrostatic and zero mass scalar fields.

It has been further shown that for a special choice of some constant parameter we get immediately the case discussed by Tiwari (1979), where the functional relation between g_{00} and the electrostatic potential ϕ is in the whole square form.

In § 4 we have given two exact interior solutions. One of them is for a spherically symmetric static dust charged in the electrical as well as in the scalar sense. The ratios of charge-to-mass densities are assumed to be constants. The solutions are found to be singular at the origin of the isotropic coordinate system. The second solution is for spherically symmetric, conformally flat, static dust charged only in the scalar sense. They are non-singular and are matched with the outside Yilmaz (1958) solution.

2. Dust distribution with electric and scalar charge in static equilibrium

Coupled Einstein–Maxwell–scalar field equations for incoherent matter containing both electric and scalar charge are

$$R^{\mu}{}_{\nu} = -8\pi(T^{\mu}{}_{\nu} - \tfrac{1}{2}\delta^{\mu}{}_{\nu}T) \tag{2.1}$$

where $T^{\mu}{}_{\nu}$ is the total energy–momentum tensor due to the dust, the electromagnetic and the scalar field. One can write $T^{\mu}{}_{\nu}$ as

$$4\pi T^{\mu}{}_{\nu} = 4\pi\rho v^{\mu}v_{\nu} + (-F^{\mu\alpha}F_{\nu\alpha} + \tfrac{1}{4}\delta^{\mu}{}_{\nu}F_{\alpha\beta}F^{\alpha\beta}) - S^{,\mu}S_{,\nu} + \tfrac{1}{2}\delta^{\mu}{}_{\nu}S^{,\alpha}S_{,\alpha}. \tag{2.2}$$

In (2.2) ρ represents the dust density, $F_{\mu\nu}$ the electromagnetic (2.2) field tensor and S the massless scalar field. Here and in what follows Greek indices stand for space–time components, Latin indices for spatial components alone. Semicolons and commas denote covariant and partial differentiations, respectively. Units are chosen so that $c = G = 1$.

We consider the line element in the form

$$ds^2 = e^{2\eta}\, dt^2 - e^{-2\eta}h_{ik}\, dx^i\, dx^k. \tag{2.3}$$

Maxwell's equations and the scalar field equations are written as

$$(\sqrt{-g}\, e^{-2\eta}g^{ik}\phi_{,k})_{,i} = 4\pi\sqrt{-g}\, e^{-\eta}\sigma, \tag{2.4}$$

$$(\sqrt{-g}\, g^{ik}S_{,k})_{,i} = 4\pi\sqrt{-g}\, \alpha, \tag{2.5}$$

where σ and α represent electric and scalar charge density respectively. We are here considering only the repulsive-type scalar field as was previously discussed by Das (1962). The static condition gives $v^i = 0$ and $v^0 = (g_{00})^{-1/2}$. Since the field is purely static we have $F_{ik} = 0$ and $F_{0i} = \phi_{,i}$, where ϕ is the electrostatic potential. Taking the divergence of (2.1) and utilising (2.2), (2.3), (2.4) and (2.5), one obtains

$$\eta_{,i} = -(\sigma/\rho)e^{-\eta}\phi_{,i} - (\alpha/\rho)S_{,i}. \tag{2.6}$$

Here if we make an assumption that S is a function of the electric potential ϕ, we immediately get a relation of the type

$$\eta_{,i} = f(x^i)\phi_{,i}. \tag{2.7}$$

Equation (2.7) shows that ϕ is functionally related to η.

We now consider the field equation

$$R^0{}_0 = -8\pi(T^0{}_0 - \tfrac{1}{2}T) \tag{2.8}$$

which gives

$$(\sqrt{-g}\, g^{ik}\eta_{,k})_{,i} + 4\pi\sqrt{-g}\,\rho - \sqrt{-g}\, e^{-2\eta} g^{ik}\phi_{,i}\phi_{,k} = 0. \tag{2.9}$$

Utilising (2.4), (2.5) and (2.6) in (2.9) one can obtain after a little manipulation the relation

$$[\sqrt{-g}\, g^{ik}\eta_{,k}\,(e^{-2\eta}\phi'^2 + S'^2 - 1)^{1/2}]_{,i} = 0. \tag{2.10}$$

Here the prime represents differentiation with respect to η. The relation (2.10) resembles equation (16) of De and Raychaudhuri (1968) and is equivalent to their relation when $S' = 0$, that is, in the absence of the scalar field. We follow now the arguments put forward by De and Raychaudhuri to write finally

$$e^{-2\eta}\phi'^2 + S'^2 - 1 = 0, \tag{2.11}$$

provided the surface of the charged dust cloud under consideration is an equipotential surface without any hole or pocket of alien matter inside.

If we write $\sigma/\rho = A$ and $\alpha/\rho = B$, equation (2.6) may be written as

$$Ae^{-\eta}\phi' + BS' = -1. \tag{2.12}$$

Combining (2.11) and (2.12), it is not difficult to get

$$e^{-\eta}\phi' = [-A \pm B(A^2 + B^2 - 1)^{1/2}]/(A^2 + B^2), \tag{2.13}$$

$$S' = [-B \pm A(A^2 + B^2 - 1)^{1/2}]/(A^2 + B^2), \tag{2.14}$$

and also

$$(Q^2 - 1)\alpha^2 + (Q\sigma - \rho)^2 = 0, \tag{2.15}$$

where we have written $e^{-\eta}\phi' = -Q$. It is interesting to note that the relation (2.15) is exactly identical to the relation (44) of Teixeira *et al* obtained for spherical symmetry and $\gamma = -1$.

For real values of $e^{-\eta}\phi'$ and S' one must have $(A^2 + B^2) \geqslant 1$, which in other words means

$$\sigma^2 + \alpha^2 \geqslant \rho^2.$$

It is interesting to note that in the absence of the scalar field ($S' = 0$) we get

$$\pm A(A^2 + B^2 - 1)^{1/2} = B$$

which can be written also after squaring as

$$(A^2 + B^2)(A^2 - 1) = 0.$$

This leads us to $A^2 = 1$ or $\rho^2 = \sigma^2$ (De and Raychaudhuri 1968). On the other hand, in the absence of the electric field ($\phi' = 0$)

$$\pm B(A^2 + B^2 - 1)^{1/2} = A,$$

which leads us to $(A^2 + B^2)(B^2 - 1) = 0$. This gives $B^2 = 1$ or $\rho^2 = \alpha^2$ (Wolk *et al* 1975).

We now proceed to show that in the interior of the singularity-free static dust charged electrically as well as in the scalar sense, h_{ik} in (2.3) must be Euclidean. The proof is as follows.

The spatial components of field equations can be written for the line element (2.3) as

$$R_{ij} = H_{ij} + 2\eta_{,i}\eta_{,j} - h^{-1/2}h_{ij}(h^{1/2}h^{km}\eta_{,k})_{,m}$$
$$= -(h_{ij}h^{km}\phi_{,k}\phi_{,m} - 2\phi_{,i}\phi_{,j})e^{-2\eta} - 4\pi\rho e^{-2\eta}h_{ij} + 2S_{,i}S_{,j} \tag{2.16}$$

for the repulsive type of scalar field, where H_{ij} is the Ricci tensor built up from h_{ij}. Again the field equation (2.8) leads to

$$-h^{-1/2}h_{ij}(h^{1/2}h^{km}\eta_{,k})_{,m} = -4\pi\rho\, e^{-2\eta}h_{ij} - e^{-2\eta}h_{ij}h^{ik}\phi_{,i}\phi_{,k}. \tag{2.17}$$

Substituting (2.17) in (2.16),

$$H_{ij} = 2(e^{-2\eta}\phi_{,i}\phi_{,j} - \eta_{,i}\eta_{,j} + S_{,i}S_{,j}). \tag{2.18}$$

Since ϕ and S are functions of η

$$H_{ij} = 2(e^{-2\eta}\phi'^2 + S'^2 - 1)\eta_{,i}\eta_{,j}. \tag{2.19}$$

In view of (2.11) one concludes

$$H_{ij} = 0, \tag{2.20}$$

which means that the three-space is flat and the line element can be written as

$$ds^2 = e^{2\eta}\, dt^2 - e^{-2\eta}\,(dx^2 + dy^2 + dz^2). \tag{2.21}$$

Again from (2.17), in view of the line element (2.21),

$$-\sum_{k=1}^{3} \eta_{,kk} = -4\pi\rho e^{-2\eta} - e^{-2\eta}\sum_{k=1}^{3}(\phi_{,k})^2. \tag{2.22}$$

In fact, the whole set of field equations now reduces to a single equation (2.22). Replacing $\phi_{,k}$ in terms of $\eta_{,k}$ and $S_{,k}$ using (2.11), we get

$$-\sum_{k=1}^{3} \eta_{,kk} = -4\pi\rho e^{-2\eta} - \sum_{k=1}^{3}(\eta_{,k})^2 + \sum_{k=1}^{3}(S_{,k})^2. \tag{2.23}$$

In the absence of the electric field $(S')^2 = (dS/d\eta)^2 = 1$ and finally (2.23) reduces to $\Sigma_{k=1}^{3} S_{,kk} = \pm 4\pi\rho \exp(\mp 2S)$, which is exactly identical to that obtained by Das (1962) for dust coupled with a scalar field. On the other hand, if the scalar field does not exist we have $S_{,k} = 0$ and from (2.23)

$$\sum_{k=1}^{3}(e^{-\eta})_{,kk} = -4\pi\rho e^{-3\eta},$$

which is again identical to the relation (28) of De and Raychaudhuri (1968) for electrically charged dust.

3. Coupled electric and scalar field in empty space

Here we consider only those situations where both g_{00} and the electrostatic potential ϕ are functions of the scalar field. We need these two assumptions separately in the exterior, while in the interior we showed previously that the functional relationship between any two of the above variables results in the functional relationship with the third also. The field equation (2.9) in this case ($\rho = 0$) gives us

$$d^2\eta/dS^2 = e^{-2\eta}\,(d\phi/dS)^2. \tag{3.1}$$

Maxwell's equation (2.4) in charge-free space ($\sigma = 0$) gives

$$(g^{ik}\sqrt{-g}\,S_{,i}S_{,k})\frac{d}{dS}\left(e^{-2\eta}\frac{d\phi}{dS}\right) = 0, \tag{3.2}$$

which reduces to

$$e^{-4\eta}\,(d\phi/dS)^2 = D^2, \tag{3.3}$$

where D is an integration constant. Combining (3.1) and (3.3), it is not difficult to get

$$(d\eta/dS)^2 = D^2 e^{2\eta} - E, \tag{3.4}$$

which can be easily integrated to determine explicitly the relationship between η and S. Combining (3.3) and (3.4), one gets in the exterior

$$e^{-2\eta}\,(d\phi/dS)^2 - (d\eta/dS)^2 = E, \tag{3.5}$$

whereas in the interior $E = -1$ in view of (2.11).

The relation (3.5) can be written also in the form

$$e^{-2\eta}\,(d\phi/d\eta)^2 - 1 = E\,(dS/d\eta)^2,$$

which again on replacing $(dS/d\eta)$ from (3.4) reduces to

$$(d\phi/d\eta)^2 = e^{2\eta}[1 + E/(D^2e^{2\eta} - E)]. \tag{3.6}$$

On integrating (3.6) one can get

$$e^{2\eta} = (\phi + F)^2 + E/D^2, \tag{3.7}$$

where F is another constant. In the special case when $E = 0$ one obtains g_{00} in the whole square form, that is

$$g_{00} = (\phi + F)^2. \tag{3.8}$$

(3.8) is exactly the case given by Tiwari (1979).

The spatial components of the field equations are now

$$R_{ij} = H_{ij} + 2\left(\frac{d\eta}{dS}\right)^2 S_{,i}S_{,j} - h^{-1/2}h_{ij}\left(h^{1/2}h^{km}\frac{d\eta}{dS}S_{,k}\right)_{,m}$$
$$= -\left[h_{ij}h^{km}\left(\frac{d\phi}{dS}\right)^2 S_{,k}S_{,m} - 2\left(\frac{d\phi}{dS}\right)^2 S_{,i}S_{,j}\right]e^{-2\eta} + 2S_{,i}S_{,j}. \tag{3.9}$$

The equation (3.9) can also be written, in view of (2.5) with $\alpha = 0$ in empty space, as

$$H_{ij} + 2\left[\left(\frac{d\eta}{dS}\right)^2 - 1 - \left(\frac{d\phi}{dS}\right)^2 e^{-2\eta}\right]S_{,i}S_{,j}$$
$$= h_{ij}h^{km}S_{,k}S_{,m}\frac{d^2\eta}{dS^2} - h_{ij}h^{km}S_{,k}S_{,m}\left(\frac{d\phi}{dS}\right)^2 e^{-2\eta}. \tag{3.10}$$

Since $(d\eta/dS)^2 = e^{-2\eta}\,(d\phi/dS)^2$, from (3.1) we finally obtain

$$H_{ij} + 2[(d\eta/dS)^2 - 1 - (d\phi/dS)^2 e^{-2\eta}]S_{,i}S_{,j} = 0, \tag{3.11}$$

which can be written in view of (3.5) as

$$H_{ij}-2(1+E)S_{,i}S_{,j}=0. \tag{3.12}$$

So if one has $E=-1$ in the empty space also, one gets $H_{ij}=0$ which means that the three-dimensional space section is Euclidean, and in that case the whole set of field equations reduces to a single equation

$$\sum_{k=1}^{3}(e^{-\eta})_{,kk}=e^{-\eta}\sum_{k=1}^{3}(S_{,k})^2. \tag{3.13}$$

Case I. In (3.12) if $E=0$, one has

$$H_{ij}-2S_{,i}S_{,j}=0. \tag{3.14}$$

This is essentially the same relation as was obtained by Tiwari (1979) except for the minus sign on the left-hand side. In Tiwari's case there was a plus sign, because he considered an attractive scalar field. The equation (3.14) represents the vacuum field equation in the absence of any electric or scalar field for the line element

$$ds^2=e^{2S}\,dt^2-e^{-2S}h_{ij}\,dx^i\,dx^j. \tag{3.15}$$

So once the solution for S is known in (3.15), it is possible to generate a vacuum solution in the presence of scalar and electric fields by integrating (3.4) for η when $E=0$. This was actually what Tiwari had done.

Case II. For $E=-1$ one can find the solution for e^{η} from (3.13), utilising the relation between η and S from (3.4). Once η is known, the metric is also known in this case because the line element is exactly identical to (2.21) valid for the interior. S and ϕ can then be determined from (3.4) and (3.6). Such a solution can perhaps match with those in the interior because the relation (3.5) in the exterior is exactly identical to (2.11) in the interior when the constant $E=-1$.

4. Some special spherically symmetric solutions

4.1. Dust with electric as well as scalar charge

The line element in (2.21) can be written in spherically symmetric form as

$$ds^2=e^{2\eta}\,dt^2-e^{-2\eta}\,(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\varphi^2), \tag{4.1}$$

where η is a function of the radial coordinate alone. The field equations are

$$-R^2{}_2=-R^3{}_3=R^0{}_0=-(\eta_{,11}+2\eta_{,1}/r)e^{2\eta}=-\phi_{,1}^2-4\pi\rho, \tag{4.2}$$

$$R^1{}_1=(\eta_{,11}-2\eta_{,1}^2+2\eta_{,1}/r)e^{2\eta}=-\phi_{,1}^2+2\gamma S_{,1}^2e^{2\eta}+4\pi\rho. \tag{4.3}$$

Here $\gamma=\pm1$ according to whether the scalar field is of repulsive or attractive type. The Maxwell and scalar equations are

$$(r^2e^{-2\eta}\phi_{,1})_{,1}=-4\pi\sigma r^2e^{-3\eta} \tag{4.4}$$

and

$$(r^2S_{,1})_{,1}=4\pi\gamma\alpha r^2e^{-2\eta} \tag{4.5}$$

and the contracted Bianchi identity is

$$\eta_{,1}+(\sigma/\rho)e^{-\eta}\phi_{,1}+(\alpha/\rho)S_{,1}=0. \tag{4.6}$$

We see that there are four equations (4.2)–(4.5) and six unknown quantities such as η, ρ, ϕ, S, σ and α. So one has the freedom to make two assumptions to get the exact solutions. The simple assumptions are perhaps

$$A = \sigma/\rho = \text{constant} \qquad \text{and} \qquad B = \alpha/\rho = \text{constant}.$$

From (4.2) and (4.3) it is easy to obtain a relation like

$$\eta_{,1}^2 = e^{-2\eta}\phi_{,1}^2 - \gamma S_{,1}^2. \tag{4.7}$$

From (4.6) and (4.7)

$$e^{-\eta}\phi_{,1} = m\eta_{,1}, \tag{4.8}$$

where m is a constant because A and B are constants. Similarly

$$S_{,1} = n\eta_{,1}, \tag{4.9}$$

where n is another constant given by a different combination of A and B and γ. Now by applying the fact that $\sigma/\alpha = A/B = K$ say, K being a constant, we get from (4.4) and (4.5) the solution for η as

$$e^{-M\eta} = (L/r + N). \tag{4.10}$$

In (4.10) L and N are constants and M is another constant given by $M = m/(m + nK)$ or in other words M is completely determined if A and B are known. The solution (4.10) is simple but unfortunately singular at $r = 0$.

4.2. *Dust with scalar charge alone*

In this context we remember an exact solution given for the interior of a spherically symmetric, conformally flat and electric charged dust by Banerjee and Som (1981). The solution was matched with the exterior Reissner–Nordström metric also. In this case we make the same assumption that the metric is conformally flat, and this assumption leads us to the exact solutions of the field equations because in the absence of the electric field we have now three equations and four unknowns (cf § 4.1.).

If we assume that the metric is conformally flat, that is the Weyl tensor $C_{\mu\nu\alpha\beta} = 0$, we get the line element (2.21) in the form

$$ds^2 = (ar^2 + b)\,dt^2 - (ar^2 + b)^{-1}(dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\varphi^2), \tag{4.11}$$

where a and b are two constants.

One should, however, remember that for the static equilibrium of scalarly charged dust the scalar field must be of repulsive character i.e. $\gamma = -1$. The mass and scalar charge density in this case are equal and are given by

$$4\pi\rho = 4\pi\alpha = (a^2r^2 + 3ab)/(ar^2 + b). \tag{4.12}$$

In order that at $r = 0$, the mass density is positive, one must have $a > 0$. Now one can match the solution (4.11) with the exterior metric given by Yilmaz (1958) for the one-parameter repulsive force, which is

$$ds^2 = e^{-2c/r}\,dt^2 - e^{2c/r}(dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\varphi^2).$$

For matching one needs at $r = r_0$

$$(g_{00})_{\text{interior}} = (g_{00})_{\text{exterior}},$$

which gives

$$ar_0^2+b=e^{-2c/r_0}. \tag{4.13}$$

The continuity of the derivative of g_{00} at $r=r_0$ further gives

$$-(2c/r_0^2)e^{2c/r_0}=2ar_0. \tag{4.14}$$

Combining (4.13) and (4.14), one finally has a and b as

$$a=(c/r_0^3)e^{-2c/r_0}, \qquad b=e^{-2c/r_0}(1-c/r_0).$$

It has already been shown that $a>0$, which leads us to the conclusion that $c>0$. Since the condition of conformal flatness only is sufficient to reduce the line element (2.21) to (4.11), it may be concluded that this metric represents the most general conformally flat solution for the interior of a static scalarly charged dust in equilibrium in an exactly analogous manner for the electrically charged dust case (Banerjee and Som 1981).

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